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LETTER TO THE EDITOR

Computation of the critical exponents of percolation

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Abstract. We compute the critical indices of percolation, using a method in which the integrals are computed at fixed dimension, at the order of two loops. Following this method we obtain the series, giving the critical exponents, expanded with respect to the coupling constant, while in the ε-expansion method the expansion of the series is performed with respect to the coupling constant and to the number of dimensions D. So we think that we have obtained better control of the position of the fixed point also at a dimension not too near to six: we are also able to improve the numerical values of the exponents at $D = 3$.

The aim of this Letter is to compute the critical indices of percolation using some techniques of field theory and a method due to Parisi (1980) (for a general review about the use of field theory methods in the field of critical phenomena see the papers by Wilson and Kogut (1974), Brézin *et al* (1976) and the book by Amit (1978)).

Following this last method, we have computed the integrals at fixed dimension, with a non-zero mass, at the order of two loops.

In this way we have obtained better results, especially for $D = 3$: in fact, resumming the series in ε, with $\varepsilon = 3$, can lead to unprecise results. Moreover, we have obtained better control of the position of the fixed point also at a dimension not too close to six. Another advantage of this method is that the series, which give the critical exponents, are just expanded with respect to a single parameter, that is, the coupling constant. In the ε-expansion method, on the other hand, there are two parameters of the expansion: the coupling constant and the dimension D. The computation of the critical exponents, using the ε-expansion method, has already been done by Amit (1976) and Priest and Lubensky (1976).

It is known that percolation is the one-state limit of the Potts model (Fortuin and Kasteleyn 1969, 1972).

Wallace and Zia (1975) have proved that the s-state Potts model, originally defined on the lattice, can be generalised to the continuum: in this limit, it generates an Euclidean field theory with $n = s - 1$ fields. Keeping only the terms which produce the strongest infrared divergences, the Lagrangian will contain only trilinear terms: this fact directly implies that the critical dimension is six.

Following Parisi's method we give the list of the definitions of the renormalised functions which appear in the Callan–Symanzyk equation and which we shall use later

to compute the critical indices:

$$C_1^*(u) = -m^2 \frac{\mathrm{d}}{\mathrm{d}m^2} \frac{\mathrm{d}}{\mathrm{d}k^2} \Gamma_{\mathrm{R}}^{(2)}(k^2)\Big|_{k^2=0}$$

$$C_2^*(u) = m^2 \frac{\mathrm{d}}{\mathrm{d}m^2} \Gamma_{\mathrm{R}}^{(2,1)}(0,0) \tag{1}$$

$$\beta^* = m^2 \frac{\mathrm{d}}{\mathrm{d}m^2} \Gamma_{\mathrm{R}}^{(3)} \qquad \beta = \beta^* + \tfrac{3}{2} u C_1^*(u)$$

$$C_2(u) = (1 - C_1^*(u)) C_2^*(u) \qquad C_1(u) = (1 - C_1^*(u)) C_1^*(u).$$

In these formulae u is the dimensionless renormalised coupling constant, and the subscript R stands for renormalised. For detailed discussions of this definition of the renormalised coupling constant see Brézin *et al* (1976).

The critical exponents are defined by

$$\eta = 2C_1(u^*) \tag{2}$$

$$\nu^{-1} = 2 + 2C_2(u^*) - \eta \tag{3}$$

where u^* is the fixed point that is the zero of the function β (equation (1)).

The functions in equation (1) can be computed using the conventional unrenormalised expansion in powers of the bare coupling constant.

At the two loops order the diagrams involved are shown in figure 1.

The contributions of the diagrams of figure 1 are composed of a part which is the value of the integral representation of the diagram itself and a part which is the symmetry factor and the tensorial coefficient (due to the continuous symmetry of the Lagrangian which defines the field theory).

The explicit form of these tensors, which are generated from the specific symmetry of the theory, is

$$Q_{ijk} = \sum_{\alpha} l_i^{\alpha} l_j^{\alpha} l_k^{\alpha}$$

where l_i^{α} is the unit vector which points in the ith direction of a hypertetrahedron in $n+1$ space, where n is the number of the components of the field and $i = 1, \ldots, n$.

The only diagrams which have a 'primitive' tensorial coupling are a_1, b_1, b_{23} (these couplings will be called α_1, β_1, β_4, respectively) (Amit 1976, Priest and Lubensky 1976).

All the other tensorial couplings can be considered as a mixture of these two; the tensorial coefficients in a_1, b_1 and b_{23} in the percolation limit are

$$\alpha_1 = -1 \qquad \beta_1 = -2 \qquad \beta_4 = 5.$$

The combinatorial coefficient for each diagram is given in table 1.

The particularity of the method we have used is that we can calculate the diagrams directly at fixed dimension after subtracting the counterterms which make them convergent at every dimension and after finding the derivatives of formulae (1).

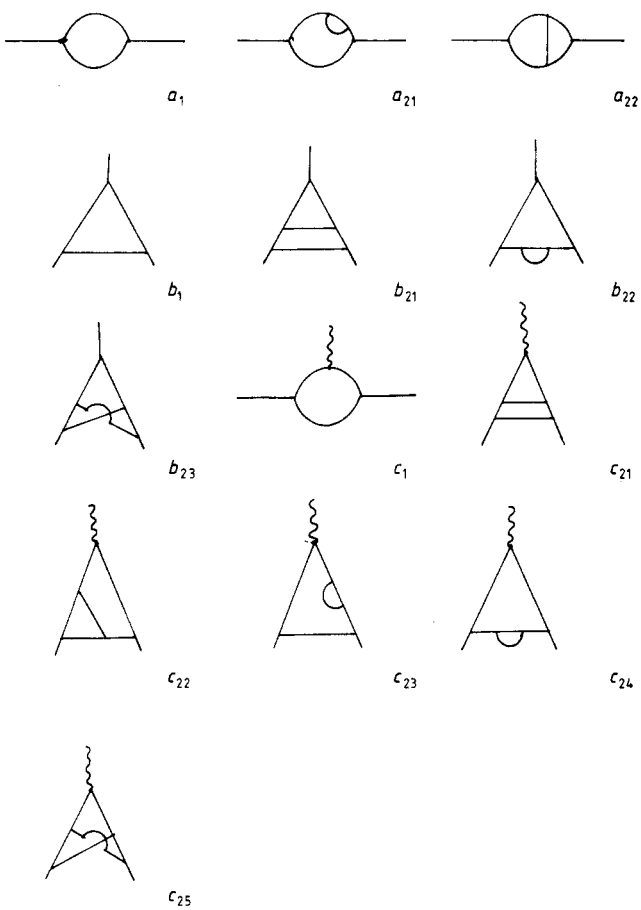

Figure 1. Diagrams involved in the computation of the critical indices at the order of two loops.

After the renormalisation procedure, the series of formulae (1) can be rewritten as

$$\beta = m^{-\varepsilon/2}\{-\tfrac{1}{4}\varepsilon u + \tfrac{1}{2}u^3\Gamma(\tfrac{1}{2}D)\Gamma(\tfrac{1}{2}(8-D))(\tfrac{1}{8}\alpha_1 - \tfrac{1}{2}\beta_1) + u^5(\tfrac{1}{2}\Gamma(\tfrac{1}{2}D))^2\Gamma(7-D)\tfrac{1}{2}[\tfrac{3}{2}\alpha_1^2 A_{21}$$
$$+\tfrac{1}{2}\beta_1\alpha_1(3A_{22}-B_{22}) - 3\beta_1^2 B_{21} - \beta_4 B_{23}]$$
$$-u^5(\Gamma(\tfrac{1}{2}D)\Gamma(\tfrac{1}{2}(8-D)))^2\tfrac{1}{48}\alpha_1(\tfrac{1}{8}\alpha_1 - \tfrac{1}{2}\beta_1)\}$$

$$C_1(u) = \frac{\alpha_1}{24}\Gamma(\tfrac{1}{2}D)\Gamma(\tfrac{1}{2}(8-D))u^2 + \tfrac{1}{2}(\tfrac{1}{2}\Gamma(\tfrac{1}{2}D))^2\Gamma(7-D)\alpha_1 u^4(\alpha_1 A_{21} + \beta_1 A_{22}) \qquad (4)$$
$$-\frac{\alpha_1^2}{(24)^2}(\Gamma(\tfrac{1}{2}D)\Gamma(\tfrac{1}{2}(8-D)))^2 u^4$$

$$C_2(u) = \tfrac{1}{2}\alpha_1\,\tfrac{1}{2}\Gamma(\tfrac{1}{2}D)\Gamma(\tfrac{1}{2}(8-D))u^2 + u^4\Gamma(7-D)(\tfrac{1}{2}\Gamma(\tfrac{1}{2}D))^2\,\tfrac{1}{2}\alpha_1$$
$$\times(\alpha_1 B_{21} + 2\beta_1 B_{21} + \tfrac{1}{2}\alpha_1 B_{22} + \beta_1 B_{23}) - \tfrac{1}{96}u^4\alpha_1^2(\Gamma(\tfrac{1}{2}D)\Gamma(\tfrac{1}{2}(8-D)))^2$$

where the contribution given by the integrals with one loop of figure 1 (after they have been derived with respect to m^2 and k^2 according to formulae (1))—a contribution that can be hand calculated—is directly expressed as the coefficient of the terms u^3 and u^2.

Table 1. List of the combinatorial coefficients of the diagrams of figure 1.

$a_1 : \frac{1}{2}\alpha_1$	$b_1 : \beta_1$	$c_1 : \alpha_1$
$a_{21} : \frac{1}{2}\alpha_1^2$	$b_{21} : 3\beta_1^2$	$c_{21} : \alpha_1^2$
$a_{22} : \frac{1}{2}\alpha_1\beta_1$	$b_{22} : \frac{3}{2}\alpha_1\beta_1$	$c_{22} : 2\alpha_1\beta_1$
	$b_{23} : \frac{1}{2}\beta_4$	$c_{23} : \alpha_1^2$
		$c_{24} : \frac{1}{2}\alpha_1^2$
		$c_{25} : \frac{1}{2}\alpha_1\beta_1$

The coefficients of u^5 and u^4 are the two loops diagrams of figure 1, derived with respect to m^2 and k^2; the parameters are dimensionless: to evaluate A_{2i} and B_{2i} we needed the help of a computer. The results are in table 2; the typical error is of unity in the last digit.

Substituting the numbers of table 2 in formulae (4) we obtain an expression for β, C_1, C_2 at each dimension.

In general, these series will be of the type

$$\beta = m^{-\varepsilon/2}(-\tfrac{1}{4}\varepsilon u + Au^3 + Bu^5) \qquad \text{or} \qquad C_i = -A_{ci}u^2 + B_{ci}u^4$$

with $i = 1, 2$.

In table 3 we report the values of A, B, A_{C_i}, B_{C_i}. After having found the fixed points (table 4) and having substituted them in (2, 3) we obtain the critical indices. The critical exponents η and ν are given in tables 4 and 5.

The exponents of table 4 are obtained by resumming the series with the Padé–Borel method (Baker *et al* 1978), while those in table 5 are obtained using the pseudo ε expansion (Le Guillou and Zinn-Justin 1980).

Finally, in table 6, we write the exponents β, γ obtained from η and ν by using scaling relations.

Our results fit fairly well with those of Kirkpatrick (1976) (table 6); as we mentioned at the beginning of this Letter we have obtained more accurate results for $D = 3$ in place of the ε-expansion method, especially for β.

Table 2. Numerical values of the coefficients A_{2i} and B_{2i} which appear in equation (4).

D	A_{21}	A_{22}	B_{21}	B_{22}	B_{23}
6	−0.0373	0.110	0.248	−0.417	0.5
5	−0.0132	0.061	0.145	−0.160	0.2331
4	-0.465×10^{-2}	0.034	0.083	−0.063	0.1147
3	-0.146×10^{-2}	0.019	0.047	−0.024	0.0582
2	-0.316×10^{-2}	0.010	0.027	−0.008	0.0301

Table 3. Coefficients at various dimensions of the series β, C_1, C_2.

D	A	B	A_{C1}	B_{C2}	A_{C2}	B_{C2}
5	2.062	2.182	0.0491	0.0218	0.2945	0.2312
4	0.875	0.617	0.0417	0.0141	0.025	0.1432
3	0.687	0.579	0.0491	0.0191	0.2945	0.1865
2	0.875	1.219	0.0833	0.0551	0.5	0.5258

Table 4. Zeros of the function β, and η and ν exponents, obtained by the Padé–Borel method.

D	u^*	η	ν
5	1.31	−0.062	0.57
4	1.65	−0.14	0.68
3	2.20	−0.23	0.84
2	3.51	−0.35	1.10

Table 5. η and ν exponents obtained by the pseudo ε expansion.

D	η	ν
5	−0.062	0.57
4	−0.14	0.69
3	−0.24	0.89
2	−0.35	1.31

Table 6. β and γ exponents obtained by scaling relations and compared with Kirkpatrick's (1976) values.

D	β	γ	β	γ
5	0.8	1.2	0.66 ± 0.05	1.3 ± 0.1
4	0.6	1.5	0.52 ± 0.03	1.6 ± 0.1
3	0.3	1.9	0.39 ± 0.02	1.8 ± 0.05
2	−0.2	2.6	0.2	2.3

The method based on the renormalisation group analysis does not succeed in determining the β index for $D = 2$: in fact, in this case, our result is in complete disagreement with Kirkpatrick's.

At the present time we are studying this. We think that this is because at $D = 2$ the fixed point becomes unstable (Amit *et al* 1977, Houghton *et al* 1978).

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